μ-Statistically Convergent Multiple Sequences in Probabilistic Normed Spaces

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Abstract In this article, we introduce the notions of μ-statistically convergent and μ-statistically Cauchy multiple sequences in probabilistic normed spaces (in short PN-spaces). We also give a suitable characterization for μ-statistically convergent multiple sequences in PN-spaces. Moreover, we introduce the notion of μ-statistical limit points for multiple sequences in PN-spaces, and we give a relation between μ-statistical limit points and limit points of multiple sequences in PN-spaces.

Keywords Probabilistic normed space · μ-statistical convergence · Multiple sequence · Two-valued measure

1 Introduction

The notion of PN-space was first introduced by Šerstnev [20] in 1963. In this theory, it has been observed that these spaces are nothing but real linear spaces where the norm of a vector is a distribution function rather than just a number. Later this theory was generalized by many authors [1, 12]. The concept of statistical convergence was first developed by Steinhaus [23] as well as by Fast [8] in 1951. Later on, this theory has been investigated by many authors in recent papers [3, 5, 9–11]. Karakus [14] has extended the concept of statistical convergence to the probabilistic normed space in 2007. In the recent past, sequence spaces have been studied by various authors [21, 26, 27] from different point of view. Moreover, Tripathy et al. [28] have studied the concepts of I-limit inferior and I-limit superior of sequences in PN-space. The notion of convergence for a sequence is also considered in measure theory. In [4], Connor has extended the concept of statistical convergence, by replacing the asymptotic density with a finitely additive two-valued measure μ. Some more work can be found in [22].

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The concepts of sequence space had been extended to double sequence by Pringsheim [17] in 1900. Then Hardy [13] introduced the concept of regular convergence for double sequence in 1917. In [14], Karakus has investigated the concept of statistical convergence in PN-spaces for single sequences. Similar concept for double sequences has been developed by Karakus and Demirci [15]. More works on statistically convergent double sequences in PN-spaces can be found in [16, 18] from different aspects. The notion of statistically convergent triple sequences defined by Orlicz function has been investigated by Datta et al. [6]. Later on, Esi and Sharma [7] have studied some paranormed sequence spaces defined by Musielak-Orlicz functions over n-normed spaces. Recently, Tripathy and Goswami [24] have introduced the notion of multiple sequences in PN-spaces, and then they have studied the statistical convergence for the same in [25]. In this paper, we investigate this concept from measure theoretic aspects.

2 Preliminaries

Throughout the paper, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ denote the sets of natural, real, and nonnegative real numbers, respectively. Moreover, μ denotes a complete $\{0, 1\}$-valued finitely additive measure defined on a field Γ of all finite subsets of $\mathbb{N}$ and suppose that $\mu(B) = 0$, if $|B| < \infty$; if $B \subset A$ and $\mu(A) = 0$, then $\mu(B) = 0$; and $\mu(\mathbb{N}) = 1$.

The definitions of distribution function and continuous t-norm can be found in [19]. Let Δ denotes the set of all distribution functions. For the definition and example of a PN-space, one may refer to [1, 2].

Definition 1 ([24]) Let $(Y, M, *)$ be a PN-space. Then, we say a multiple sequence $y = (y_{k_1k_2\ldots k_n})$ is convergent to $\xi \in Y$ in terms of probabilistic norm M, if for every $\delta > 0$ and $\gamma \in (0, 1)$, there is an $n_0 \in \mathbb{N}$ such that $M_{y_{k_1k_2\ldots k_n}-\xi}(\delta) > 1 - \gamma$, for all $k_i \geq n_0$, for $i = 1, 2, \ldots, n$. It is denoted by $M - \lim y_{k_1k_2\ldots k_n} = \xi$.

Definition 2 ([24]) Let $(Y, M, *)$ be a PN-space. Then, we say a multiple sequence $y = (y_{k_1k_2\ldots k_n})$ is Cauchy in terms of probabilistic norm M, if for every $\delta > 0$ and $\gamma \in (0, 1)$, there is an $n_0 \in \mathbb{N}$ such that $M_{y_{k_1k_2\ldots k_n}-y_{m_1m_2\ldots m_n}}(\delta) > 1 - \gamma$, for all $k_i \geq n_0$ and $m_i \geq n_0$, for $i = 1, 2, \ldots, n$.

3 μ-Statistically Convergent Multiple Sequences in PN-Space

In this section, we introduce the following definitions and give some useful characterizations for μ-statistical convergence of multiple sequence in PN-spaces.

Definition 3 A multiple sequence $y = (y_{k_1 k_2 \ldots k_n})$ in a PN-space $(Y, M, *)$ is said to be μ-statistically null in terms of the probabilistic norm M, if for every $\delta > 0$ and $\gamma \in (0, 1)$, we have

$$\mu\left(\left\{(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n : M_{y_{k_1 k_2 \ldots k_n}}(\delta) \le 1 - \gamma\right\}\right) = 0.$$

Definition 4 A multiple sequence $y = (y_{k_1 k_2 \ldots k_n})$ in a PN-space $(Y, M, *)$ is said to be μ-statistically bounded in terms of probabilistic norm M, if there exists an $\delta > 0$ such that

$$\mu\left(\left\{(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n : M_{y_{k_1 k_2 \ldots k_n}}(\delta) \le 1 - \gamma\right\}\right) = 0, \text{ for every } \gamma \in (0, 1).$$

Definition 5 A multiple sequence $y = (y_{k_1 k_2 \ldots k_n})$ in a PN-space $(Y, M, *)$ is said to be μ-statistically convergent to $\xi \in Y$ in terms of the probabilistic norm M, if for every $\delta > 0$ and $\gamma \in (0, 1)$, we have

$$\mu\left(\left\{(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n : M_{y_{k_1 k_2 \ldots k_n} - \xi}(\delta) \le 1 - \gamma\right\}\right) = 0,$$

and we write as $\mu - stat_M - \lim y_{k_1 k_2 \ldots k_n} = \xi$.

Definition 6 A multiple sequence $y = (y_{k_1 k_2 \ldots k_n})$ in a PN-space $(Y, M, *)$ is called μ-statistically Cauchy in terms of probabilistic norm M, if for every $\delta > 0$ and $\gamma \in (0, 1)$, there is an $n_0 \in \mathbb{N}$ such that

$$\mu\left(\left\{(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n : M_{y_{k_1 k_2 \ldots k_n} - y_{m_1 m_2 \ldots m_n}}(\delta) \le 1 - \gamma\right\}\right) = 0.$$

From the above definitions, we have the following two results. The proofs are obvious, so omitted.

Theorem 1 *Let $(Y, M, *)$ be a probabilistic normed space. Then, for every $\gamma \in (0, 1)$ and $\delta > 0$, the following statements are equivalent:*

1. $\mu - stat_M - \lim y_{k_1 k_2 \ldots k_n} = \xi$.
2. $\mu\left(\left\{(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n : M_{y_{k_1 k_2 \ldots k_n} - \xi}(\delta) \le 1 - \gamma\right\}\right) = 0$.
3. $\mu\left(\left\{(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n : M_{y_{k_1 k_2 \ldots k_n} - \xi}(\delta) > 1 - \gamma\right\}\right) = 1$.
4. $\mu - stat - \lim M_{y_{k_1 k_2 \ldots k_k} - \xi}(\delta) = 1$.

Corollary 1 *Let $(Y, M, *)$ be a PN-space. If a multiple sequence $y = (y_{k_1 k_2 \ldots k_n})$ in $(Y, M, *)$ is μ-statistically convergent in terms of probabilistic norm M, then $\mu - stat_M - \lim y$ is unique.*

Corollary 2 *Let $(Y, M, *)$ be a probabilistic normed space. If $M - \lim y_{k_1 k_2 \ldots k_n} = \xi$, then $\mu - stat_M - \lim y_{k_1 k_2 \ldots k_n} = \xi$.*

Proof Suppose $y = (y_{k_1k_2\dots k_n})$ converges to ξ in terms of probabilistic norm M. Then, for every $\delta > 0$ and $\gamma \in (0, 1)$, there exists an $n_0 \in \mathbb{N}$ such that

$$M_{y_{k_1k_2\dots k_n}-\xi}(\delta) > 1-\gamma, \quad \text{for all } k_i \geq n_0,\ i = 1, 2, \dots, n.$$

Then, the set $\left\{(k_1, k_2, \dots, k_n) \in \mathbb{N}^n : M_{y_{k_1k_2\dots k_n}-\xi}(\delta) \leq 1-\gamma\right\}$ contains at most finite numbers of terms, and so we have

$$\mu\left(\left\{(k_1, k_2, \dots, k_n) \in \mathbb{N}^n : M_{y_{k_1k_2\dots k_n}-\xi}(\delta) \leq 1-\gamma\right\}\right) = 0.$$

Consequently, $\mu - stat_M - \lim y_{k_1k_2\dots k_n} = \xi$.

The converse of the Corollary 2 does not hold, in general.

Example 1 Suppose $(\mathbb{R}, ||\cdot||)$ is the space of all real numbers with the standard norm. Let $a_1 * a_2 = a_1a_2$ and $M_y(s) = \frac{s}{s+||y||}$, where $y \in R$ and $s \geq 0$. Then, we see that $(\mathbb{R}, M, *)$ is a probabilistic normed space. Let $K \subset \mathbb{N}^n$ be such that $\mu(K) = 0$. We define a sequence $y = (y_{k_1k_2\dots k_n})$ as follows:

$$y_{k_1k_2\dots k_n} = \begin{cases} k_1k_2\dots k_n, & \text{if } (k_1, k_2, \dots, k_n) \in K \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Then, one can easily verify that $y = (y_{k_1k_2\dots k_n})$ is μ-statistically convergent in terms of the probabilistic norm M. However, the sequence $y = (y_{k_1k_2\dots k_n})$ defined by (1) is not convergent in the space $(\mathbb{R}, ||\cdot||)$, thus we conclude that y is also not convergent in terms of the probabilistic norm M.

Theorem 2 *Suppose that $y = (y_{k_1k_2\dots k_n})$ is a multiple sequence in a probabilistic normed space $(Y, M, *)$. Then $\mu - stat_M - \lim y_{k_1k_2\dots k_n} = \xi$ if and only if there is an index subset $A = \left\{(n_{k_1}, n_{k_2}, \dots, n_{k_n}) : n_{k_i} \in \mathbb{N}\right\}$ of $\mathbb{N}^n$ such that $\mu(A) = 1$ and*

$$M - \lim_{(k_1,k_2,\dots,k_n)\in A} y_{k_1k_2\dots k_n} = \xi.$$

Proof First, suppose that $\mu - stat_M - \lim y_{k_1k_2\dots k_n} = \xi$. Then, for every $\delta > 0$ and $s \in \mathbb{N}$, we define the following two sets:

$$A(s, \delta) = \left\{(k_1, k_2, \dots, k_n) \in \mathbb{N}^n : M_{y_{k_1k_2\dots k_n}-\xi}(\delta) \leq 1 - \frac{1}{s}\right\} \tag{2}$$

$$B(s, \delta) = \left\{(k_1, k_2, \dots, k_n) \in \mathbb{N}^n : M_{y_{k_1k_2\dots k_n}-\xi}(\delta) > 1 - \frac{1}{s}\right\}. \tag{3}$$

Then, we have $\mu\left(A(s, \delta)\right) = 0$ and

$$B(1, \delta) \supset B(2, \delta) \supset \cdots \supset B(j, \delta) \supset B(j+1, \delta) \supset \dots \tag{4}$$

$$\mu(B(s, \delta)) = 1, \quad \text{for } s = 1, 2, \dots. \tag{5}$$

Now, we need to show that, the sequence $y = (y_{k_1 k_2 \dots k_n})$ is convergent to ξ in terms of probabilistic norm M, for $(k_1, k_2, \dots, k_n) \in B(s, \delta)$. If possible, suppose that $y = (y_{k_1 k_2 \dots k_n})$ is not convergent to ξ in terms of the probabilistic norm M. Then, there exists $\gamma > 0$ such that the set

$$\left\{(k_1, k_2, \dots, k_n) \in \mathbb{N}^n : M_{y_{k_1 k_2 \dots k_n} - \xi}(\delta) \le 1 - \gamma\right\}$$

contains infinite number of terms. Let

$$B(\gamma, \delta) = \left\{(k_1, k_2, \dots, k_n) \in \mathbb{N}^n : M_{y_{k_1 k_2 \dots k_n} - \xi}(\delta) > 1 - \gamma\right\},$$

where $\gamma > \frac{1}{s}$, for $s = 1, 2, \dots$. Then $\mu\left(B(\gamma, \delta)\right) = 0$. But from (4), we have $B(s, \delta) \subset B(\gamma, \delta)$. Thus, we obtain $\mu(B(s, \delta)) = 0$, which is a contradiction to (5). Hence $y = (y_{k_1 k_2 \dots k_n})$ is convergent to ξ in terms of the probabilistic norm M.

Conversely, we assume that there is an index subset $A = \{(k_1, k_2, \dots, k_n) : k_i \in \mathbb{N}\} \subset \mathbb{N}^n$ such that $\mu(A) = 1$ and

$$N - \lim_{(k_1, k_2, \dots, k_n) \in A} y_{k_1 k_2 \dots k_n} = \xi.$$

Then, for every $\delta > 0$ and $\gamma \in (0, 1)$, there is an $m_0 \in \mathbb{N}$ such that

$$M_{y_{k_1 k_2 \dots k_n} - \xi}(\delta) > 1 - \gamma, \quad \text{for } k_i \ge m_0, \ i = 1, 2, \dots, n.$$

Now, we see that

$$\{(k_1, k_2, \dots, k_n) \in \mathbb{N}^n : M_{y_{k_1 k_2 \dots k_n} - \xi}(\delta) \le 1 - \gamma\}$$
$$\subset \mathbb{N}^n - \{(k_{1(m_0+1)}, \dots, k_{n(m_0+1)}), (k_{1(m_0+2)}, \dots, k_{n(m_0+2)}), \dots\}.$$

Therefore, we have $\mu\left(\left\{(k_1, k_2, \dots, k_n) \in \mathbb{N}^n : M_{y_{k_1 k_2 \dots k_n} - \xi}(\delta) \le 1 - \gamma\right\}\right) \le 1 - 1 = 0$. Consequently, we have $\mu - stat_M - \lim y_{k_1 k_2 \dots k_n} = \xi$.

Theorem 3 *Let $y = (y_{k_1 k_2 \dots k_n})$ be a multiple sequence in a PN-space $(Y, M, *)$. Then the following statements are equivalent:*

1. *y is a μ-statistically Cauchy sequence in terms of probabilistic norm M.*
2. *There is an index subset $A = \left\{(m_{k_1}, m_{k_2}, \dots, m_{k_n}) \in \mathbb{N}^n : m_{k_i} \in \mathbb{N}\right\} \subset \mathbb{N}^n$ such that $\mu(A) = 1$ and the subsequence $\left\{y_{m_{k_1} m_{k_2} \dots m_{k_n}}\right\}_{(m_{k_1}, m_{k_2}, \dots, m_{k_n}) \in A}$ is a Cauchy sequence in terms of the probabilistic norm M.*

Proof The proof is easy and so omitted.

We now give some arithmetical properties of μ-statistical convergence for a multiple sequence on PN-space.

Theorem 4 *Let* $(Y, M, *)$ *be a probabilistic normed space. Then*

1. *If* $\mu - stat_M - \lim x_{k_1k_2\ldots k_n} = \alpha$ *and* $\mu - stat_M - \lim y_{k_1k_2\ldots k_n} = \beta$*, then* $\mu - stat_M - \lim(x_{k_1k_2\ldots k_n} + y_{k_1k_2\ldots k_n}) = \alpha + \beta$.
2. *If* $\mu - stat_M - \lim x_{k_1k_2\ldots k_n} = \alpha$ *and* $a \in \mathbb{R}$*, then* $\mu - stat_M - \lim ax_{k_1k_2\ldots k_n} = a\alpha$.
3. *If* $\mu - stat_M - \lim x_{k_1k_2\ldots k_n} = \alpha$ *and* $\mu - stat_M - \lim y_{k_1k_2\ldots k_n} = \beta$*, then* $\mu - stat_M - \lim(x_{k_1k_2\ldots k_n} - y_{k_1k_2\ldots k_n}) = \alpha - \beta$.

Proof The proof follows from the definition of μ-statistical convergence of a multiple sequence in PN-space itself.

4 μ-Statistical Limit Points for Multiple Sequences in PN-Space

In this section, we introduce the concepts of μ-statistical limit points of multiple sequences in PN-spaces and investigate their relation with limit points of multiple sequences in PN-spaces.

Definition 7 ([24]) Let $(Y, M, *)$ be a probabilistic normed space, and let $y = (y_{k_1k_2\ldots k_n})$ be a multiple sequence. We say that $\xi \in Y$ is a limit point of y in terms of the probabilistic norm M, if there is a subsequence of y that converge to ξ in terms of the probabilistic norm M. Let $L_M(y)$ denotes the set of all limit points of the multiple sequence $y = (y_{k_1k_2\ldots k_n})$.

Definition 8 Let $(Y, M, *)$ be a probabilistic normed space, and let $y = (y_{k_1k_2\ldots k_n})$ be a multiple sequence. We say that $\eta \in Y$ is a μ-statistical limit point of the multiple sequence y in terms of the probabilistic norm M, if there is a set

$$A = \{(k_1(i), k_2(i), \ldots, k_n(i)) : k_j(1) < k_j(2) < k_j(3) < \ldots, \text{ for } j = 1, 2, \ldots, n\} \subset \mathbb{N}^n$$

such that $\mu(A) \neq 0$ and $M - \lim y_{k_1(i)k_2(i)\ldots k_n(i)} = \eta$. Let $\Lambda_M^\mu(y)$ denote the set of all $\mu - stat_M - limit$ points of the multiple sequence $y = (y_{k_1k_2\ldots k_n})$.

Theorem 5 *Suppose* $y = (y_{k_1k_2\ldots k_n})$ *is a multiple sequence in a PN-space* $(Y, M, *)$*. If* $\mu - stat_M - \lim y = L_1$*, then* $\Lambda_M^\mu(y) = \{L_1\}$.

Proof If possible, suppose that $\Lambda_M^\mu(y) = \{L_1, L_2\}$ such that $L_1 \neq L_2$. Then there exists two sets:

$$A = \{(k_1(i), k_2(i), \ldots, k_n(i)) : k_j(1) < k_j(2) < k_j(3) < \ldots, \text{ for } j = 1, 2, \ldots, n\} \subset \mathbb{N}^n$$

$$B = \{(u_1(i), u_2(i), \ldots, u_n(i)) : u_j(1) < u_j(2) < u_j(3) < \ldots, \text{ for } j = 1, 2, \ldots, n\} \subset \mathbb{N}^n$$

such that $\mu(A) \neq 0$, $\mu(B) \neq 0$ and $M - \lim y_{k_1(i)k_2(i)\ldots k_n(i)} = L_1$, $M - \lim y_{u_1(i)u_2(i)\ldots u_n(i)} = L_2$. Since $M - \lim y_{u_1(i)u_2(i)\ldots u_n(i)} = L_2$, so for every $\delta > 0$ and $\gamma \in (0, 1)$, we have

$$\mu\left(\left\{(u_1(i), u_2(i), \ldots, u_n(i)) \in \mathbb{N}^n : M_{y_{u_1(i)u_2(i)\ldots u_n(i)} - L_2}(\delta) \leq 1 - \gamma\right\}\right) = 0.$$

Now, we see that

$$\left\{(u_1(i), u_2(i), \ldots, u_n(i)) \in \mathbb{N}^n : i \in \mathbb{N}\right\}$$
$$= \left\{(u_1(i), u_2(i), \ldots, u_n(i)) \in \mathbb{N}^n : M_{y_{u_1(i)u_2(i)\ldots u_n(i)} - L_2}(\delta) > 1-\gamma\right\}$$
$$\cup \left\{(u_1(i), u_2(i), \ldots, u_n(i)) \in \mathbb{N}^n : M_{y_{u_1(i)u_2(i)\ldots u_n(i)} - L_2}(\delta) \leq 1-\gamma\right\}$$

which implies that

$$\mu\left(\left\{(u_1(i), u_2(i), \ldots, u_n(i)) \in \mathbb{N}^n : M_{y_{u_1(i)u_2(i)\ldots u_n(i)} - L_2}(\delta) > 1-\gamma\right\}\right) \neq 0. \tag{6}$$

However $\mu - stat_M - \lim y = L_1$ implies that for every $\delta > 0$,

$$\mu\left(\left\{(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n : M_{y_{k_1k_2\ldots k_n} - L_1}(\delta) \leq 1-\gamma\right\}\right) = 0. \tag{7}$$

Thus, we can write $\mu\left(\left\{(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n : M_{y_{k_1k_2\ldots k_n} - L_1}(\delta) > 1-\gamma\right\}\right) \neq 0$. Now, for every $L_1 \neq L_2$, we have

$$\left\{(u_1(i), u_2(i), \ldots, u_n(i)) \in \mathbb{N}^n : M_{y_{u_1(i)u_2(i)\ldots u_n(i)} - L_2}(\delta) > 1-\gamma\right\}$$
$$\cap \left\{(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n : M_{y_{k_1k_2\ldots k_n} - L_1}(\delta) > 1-\gamma\right\} = \phi.$$

Therefore

$$\left\{(u_1(i), u_2(i), \ldots, u_n(i)) \in \mathbb{N}^n : M_{y_{u_1(i)u_2(i)\ldots u_n(i)} - L_2}(\delta) > 1-\gamma\right\}$$
$$\subseteq \left\{(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n : M_{y_{k_1k_2\ldots k_n} - L_1}(\delta) \leq 1-\gamma\right\},$$

which implies that $\mu\left(\left\{(u_1(i), u_2(i), \ldots, u_n(i)) \in \mathbb{N}^n : M_{y_{u_1(i)u_2(i)\ldots u_n(i)} - L_2}(\delta) > 1-\gamma\right\}\right) = 0$. This contradicts the Eq. (6). Hence, we must have $\Lambda_M^{\mu}(y) = \{L_1\}$.

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